Math 451: Introduction to General Topology
Lecture 4

Metric spaces.

Def. Let X be a set. A metric (distance function) on X is a map $d : X \times X \to [0,\infty)$ such that
(i) $d(x,x) = 0$ for all $x \in X$ and $d(x,y) = 0 \Rightarrow x = y$ for all $x, y \in X$;
(ii) (symmetry) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii) (triangle inequality) $d(x,z) \le d(x,y) + d(y,z)$ for all $x,y,z \in X$.
A set X equipped with a metric d is called a metric space and denoted (X,d).

Def. Let (X,d) be a metric space. For $x \in X$ and $r \ge 0$, we define the (open) ball at x of radius r to be

$$B_r(x) := \{y \in X : d(x,y) < r\}.$$

We also define the closed ball at x of radius r as $\bar{B}_r(x) := \{y \in X : d(x,y) \le r\}$.
By a ball in X we mean an open ball $B_r(x)$ for some $x \in X$ and $r \ge 0$.

Examples and nonexamples.

(a) $\mathbb{R}$ with its usual distance function $d(x,y) := |y-x|$ is a metric space.
Balls in this metric space are exactly the bounded open intervals (a,b). Indeed, $(a,b) = B_{\frac{b-a}{2}}\left(\frac{b+a}{2}\right)$ and for each $x \in X$ and $r \ge 0$, $B_r(x) = (x-r, x+r)$.

(b) For any set X, the function $d(x,y) := \begin{cases} 1 & \text{if } x \ne y \\ 0 & \text{if } x = y \end{cases}$ defines a metric on X, called the 0-1 metric. In this metric, for every $x \in X$ and $r > 0$,

$$B_r(x) = \begin{cases} \{x\} & \text{if } r \le 1 \\ X & \text{otherwise} \end{cases}.$$

(c) For $\mathbb{R}$ define $d_{\frac{1}{2}}(x,y) := \sqrt{|y-x|}$. This is a metric.

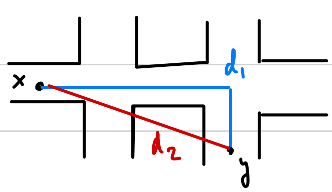
Proof. For $x,y,z\in\mathbb{R}$, we want to show $\sqrt{|x-z|}\le\sqrt{|x-y|}+\sqrt{|y-z|}$, which is equivalent to
$$\left(\sqrt{|x-y|}+\sqrt{|y-z|}\right)^2 \geq |x-z|.$$

But $\left(\sqrt{|x-y|}+\sqrt{|y-z|}\right)^2 = |x-y|+|y-z|+2\sqrt{|x-y||y-z|} \geq |x-y|+|y-z| \geq |x-z|.$ □

(d) For $\mathbb{R}$, the function $d_2(x,y):=|y-x|^2$ is not a metric: take $x<y<z$
$$d_2(x,z)=(z-x)^2=(z-y+y-x)^2=(z-y)^2+(y-x)^2+2(z-y)(y-x)>(z-y)^2+(y-x)^2=d_2(y,z)+d_2(x,y).$$
Hence the Δ-inequality fails.

Fact. Let $\varphi:[0,\infty)\to[0,\infty)$ be a strictly increasing function and define $d_\varphi:\mathbb{R}\to[0,\infty)$ by $d_\varphi(x,y):=\varphi(|y-x|)$. If φ is concave (e.g. $\varphi=\sqrt{\cdot}$) then d_φ is a metric, while if φ is convex (e.g. $\varphi=(\cdot)^2$) then d_φ is not a metric.

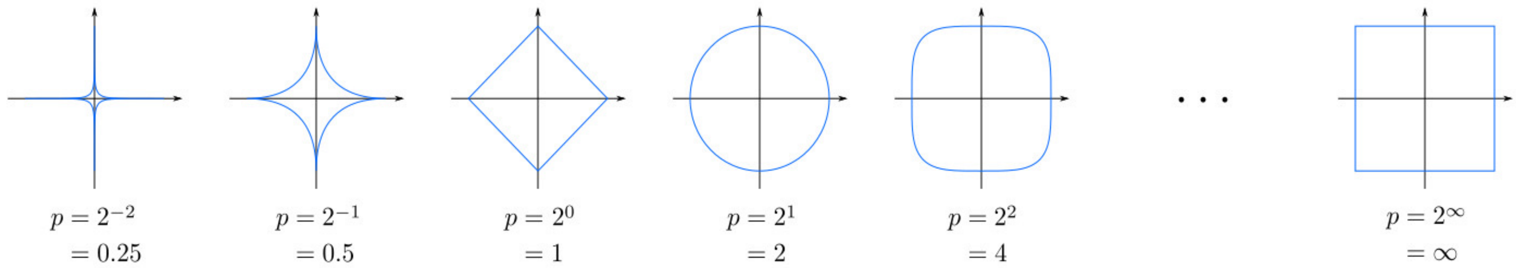
(e) For the d-dimensional Euclidean space $\mathbb{R}^d$, there are various natural metrics on it. Fix $1\le p<\infty$. Then define $d_p:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ by
$$d_p(x,y):=\left(\sum_{i=1}^d |x_i-y_i|^p\right)^{\frac{1}{p}}.$$
This indeed satisfies the Δ-inequality (HW), and it is call the p-metric.
For $p=2$, d_2 is called the Euclidean metric,
$p=1$, d_1 is often called the New York distance because, in $d=2$,



$$d_1(x,y)=|x_1-y_1|+|x_2-y_2|$$

We can also define d_p for $p=\infty$: put $d_\infty(x,y):=\max\limits_{1\le i\le d}|x_i-y_i|$.

To understand the relationship between these metrics, let's look at the unit balls at 0 as p varies (for $d=2$):



As the pictures suggest, when $p \to \infty$ the metric d_p tends to d_∞. Indeed this is true (HW):

$$\lim_{p \to \infty} d_p = d_\infty.$$

Furthermore, all these d_p metrics are Lipschitz equivalent.

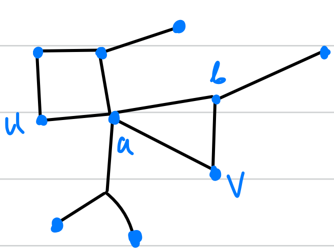
Def. Let X be a set. Two metrics d_1, d_2 on X are called Lipschitz equivalent if there are constants $C_1, C_2 > 0$ such that

$$C_1 d_2 \le d_1 \le C_2 d_2.$$

Prop. For any $1 \le p, q \le \infty$, d_p and d_q are Lipschitz equivalent.

We leave the proof as a HW exercise.

(f) Let $G := (V, E)$ be an undirected connected graph with a vertex set V and edge set E.



The graph distance is the function $d_G : V \times V \to \mathbb{N}$ defined by

$$d_G(u,v) := \text{the length of a shortest path from } u \text{ to } v.$$

In this example, $d_G(u,v) = 2$ because of the path $u-a-v$. (Although there is a longer path $u-a-b-v$.)

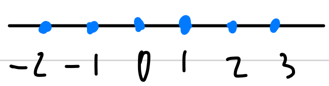
It is obvious that d_G satisfies the Δ-inequality and is hence a metric. Here too, as with the 0-1 metric, singletons are balls of radius, say, $\frac{1}{2}$.

(g) Let G be group, for example $G = \mathbb{Z}$ or $\mathbb{Z}^d$ or the free group $\mathbb{F}_d$ on d generators. Suppose G is finitely generated and let S be a finite symmetric set of generators, i.e. if $s \in S$ then $s^{-1} \in S$. For example, for $\mathbb{Z}$, we can take $S = \{1, -1\}$, and for $\mathbb{Z}^2$, $S = \{\pm(1,0), \pm(0,1)\}$. For $\mathbb{F}_2 = \langle a, b \rangle$, we take $S = \{a^{\pm 1}, b^{\pm 1}\}$.

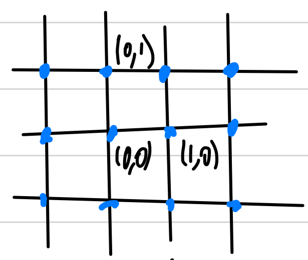
We define a graph with the vertex set G itself and with the edge set E defined by:

$$(g, h) \in E \iff h = gs \text{ for some } s \in S.$$

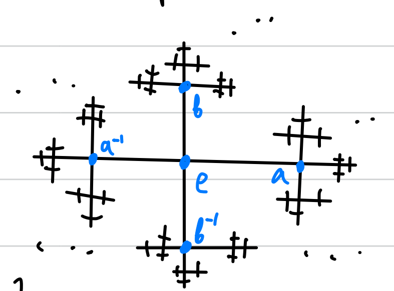
We call this graph the Cayley graph of G with respect to S and denote it $\mathrm{Cay}_S(G)$.



$\mathrm{Cay}_S(\mathbb{Z})$, $S = \{\pm 1\}$



$\mathrm{Cay}_S(\mathbb{Z}^2)$, $S = \{\pm(1,0), \pm(0,1)\}$



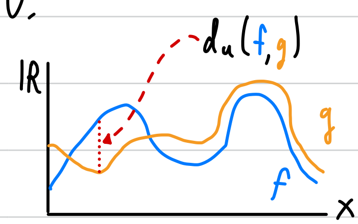
$\mathrm{Cay}_S(\mathbb{F}_2)$, $S = \{a^{\pm 1}, b^{\pm 1}\}$.

The graph distance on the $\mathrm{Cay}_S(G)$ turns the group G into metric space, opening another — geometric — avenue for studying the group. The subject that does this is called geometric group theory.

(h) Let X be a set. Denote by $B(X)$ the set of all bounded real-valued functions on X. We recall that $f: X \to \mathbb{R}$ is called bounded if $f(X)$ is a bounded subset of $\mathbb{R}$, i.e. $f(X) \subseteq (-r, r) = B_r(0)$ for some $r > 0$.

Let $d_u : B(X) \times B(X) \to [0, \infty)$ be defined by

$$d_u(f,g) := \sup_{x \in X} |f(x) - g(x)|$$



Note that $B(X)$ is closed under finite linear combinations, i.e. $B(X)$ is a vector space over $\mathbb{R}$. In particular, if $f, g \in B(X)$, then $f+g$ and $f-g$ are also bdd, so $d_u(f,g) < \infty$.

Claim. d_u is a metric, called the uniform metric.

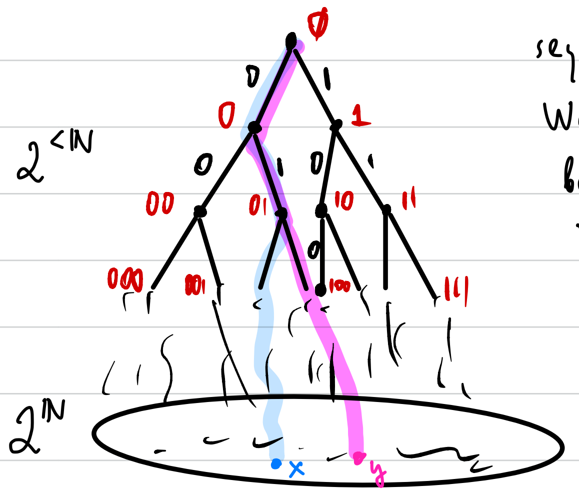
Proof. For Δ-inequality, observe that for all $f, g, h \in B(X)$ and $x \in X$,

$$|f(x) - h(x)| = |f(x) - g(x) + g(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)| \leq d_u(f, g) + d_u(g, h).$$

Since this holds for all $x \in X$, we get:

$$d_u(f, h) = \sup_{x \in X} |f(x) - h(x)| \leq d_u(f, g) + d_u(g, h).$$ □

(g) Let Σ be a ctbl nonempty set, e.g. $\Sigma = 2 = \{0, 1\}$ or $\Sigma = \mathbb{N}$.
Let $X := \Sigma^{\mathbb{N}} := \{(x_n)_{n \in \mathbb{N}} : x_n \in \Sigma \text{ for all } n \in \mathbb{N}\}$.
So for example, $2^{\mathbb{N}}$ is the set of 0-1 sequences and $\mathbb{N}^{\mathbb{N}}$ is the set of sequences of natural numbers. To picture this we draw the set $\Sigma^{<\mathbb{N}}$ of all finite sequences in Σ as a tree, depicted on the left for $\Sigma = 2$. We then think of the elements of $\Sigma^{\mathbb{N}}$ as the infinite branches through this tree.



We define a metric on $\Sigma^{\mathbb{N}}$ as follows: for $x \neq y \in \Sigma^{\mathbb{N}}$

$$d(x, y) := 2^{-\Delta(x,y)},$$

where $\Delta(x, y) :=$ the least index $n \in \mathbb{N}$ with $x(n) \neq y(n)$.
Of course we set $d(x, y) := 0$ if $x = y$.